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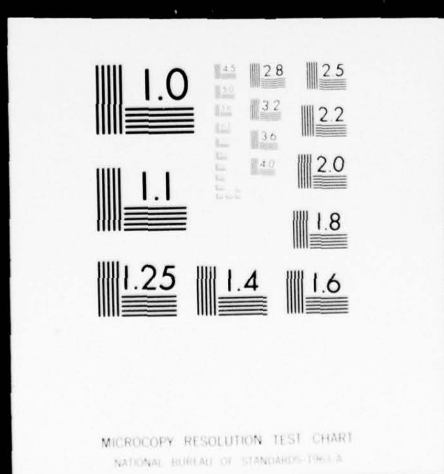

LOCAL MAXIMA OF THE SAMPLE FUNCTIONS OF THE N-PARAMETER BESSEL PROCESS

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Abstract. In this paper we show that almost every sample function of the N-parameter Bessel process associated with the N-parameter Wiener process has a local maxima. In addition some properties related to the local maxima are investigated.

1. Introduction and Preliminaries. Let $W^{(N)}$ be the N-parameter Wiener process, that is a real valued Gaussian process with zero means and covariance $\prod_{i=1}^{N}(s_i \wedge t_i)$ where $s = \langle s_i \rangle$, $t = \langle t_i \rangle$, $s_i \geq 0$, $t_i \geq 0$, $i = 1, \ldots, N$. Then $W^{(N,d)}$ is to be the process with values in d-dimensional Euclidean space R^d such that each component is an

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N-parameter Wiener process, the components being independent. Write $W = W^{(N,d)}$ for simplicity, and denote the ith component of W by W^i. Define the N-parameter Bessel process associated with W by

$$(1.1) \qquad B_t = \left[\sum_{i=1}^{d} (W_t^i)^2\right]^{\frac{1}{2}}$$

It is shown that almost every sample function of B_t has a local maxima. Furthermore some properties related to the local maxima of B_t are investigated.

As in Orey and Pruitt** (1973), our parameter space is R_+^N, that is the set of $t \in R^N$ with all components non-negative. When dealing with a point t in the parameter space we sometimes write $t = \langle t_1, \ldots, t_N\rangle$ or simply $\langle t_i\rangle$. In case all $t_i = 0$, we write $t = \langle 0\rangle$. For $s = \langle s_i\rangle$ and $t = \langle t_i\rangle$ with $s_i \le t_i$, the interval $\mathop{X}_{i=1}^{N}[s_i, t_i]$ is denoted by $\Delta(s,t)$, and by $\Delta(t)$ in case $s = \langle 0\rangle$. Denote by $S(s,t)$, the symmetric difference of $\Delta(s)$ and $\Delta(t)$. Then it is easy to check that if $s,t \in R_+^N$, the variance of $W^i(t) - W^i(s)$ is $|S(s,t)|$ where $|\cdot|$ denotes the N-dimensional Lebesgue measure. Furthermore,

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** Wherever possible, we shall use the notation of Orey and Pruitt (1973).

W has continuous sample functions and independent increments. We denote the increment of W over $\Delta(s,t)$ by $W(\Delta(s,t))$. For further information on W, the reader is referred to Kitagawa (1951), Chentsov (1956), Yeh (1960, 1963a, 1963b), Park, C. (1969), Park, W. J. (1970), Zimmerman (1972) and Orey and Pruitt (1973).

Definition 1.1. The sample function $B(\cdot,\omega)$ has a local maxima at s if there exists an open set 0 containing s such that $0 \subset R_+^N$ and $B(t,\omega) \leq B(s,\omega)$ for all $t \in 0$.

We shall need the Orey-Pruitt analogue of the familiar zero-one law. Let C_n be the class of time intervals in R_+^N with vertices of the form $\langle k_i 2^{-n} \rangle$, k_i nonnegative integers, and having all sides of equal length, and for $n > 0$ each member of C_n is to be a subcube of one in C_0. Let $C_\infty = \bigcup_{n=0}^{\infty} C_n$, and $\mathfrak{F}_n = \mathcal{B}(W(\Delta), \Delta \in C_n)$, $\mathfrak{F}_\infty = \bigvee_{n=0}^{\infty} \mathfrak{F}_n$. Thus $\mathfrak{F}_n$ is the Borel field generated by the indicated class of random variables and $\mathfrak{F}_\infty$ is the smallest Borel field including all $\mathfrak{F}_n$. For a subset D of R_+^N, we put $C_n(D) = \{\Delta \in C_n ; \Delta \subseteq D\}$, $\mathfrak{F}_n(D) = \mathcal{B}\{W(\Delta) ; \Delta \in C_n(D)\}$, $\mathfrak{F}_\infty(D) = \bigvee_{n=0}^{\infty} \mathfrak{F}_n(D)$ then we have the following lemma.

Lemma 1.1 (Orey-Pruitt (1973)). Let $D_m \subseteq R_+^N$, $m = 1, 2, \ldots,$

with $D_m \downarrow \phi$. *If* $A \in \mathcal{F}_\infty(D_m)$ *for every* m , *then* $P(A) = \{0,1\}$.

Lemma 1.2. *Let* φ *be a nonnegative, nondecreasing, continuous function defined for large arguments. Then for almost all* ω *there is an* $\epsilon(\omega)$ *such that for all intervals* $\Delta(s,t)$ *with* $\Delta(s,t) \subset \Delta(\langle 1 \rangle)$ *and* $|\Delta(s,t)| < \epsilon(\omega)$,

$$|W(\Delta(s,t))| < |\Delta(s,t)|^{\frac{1}{2}}\varphi(|\Delta(s,t)|^{-1})$$

if and only if

$$\int^\infty \log \xi^{3N + d/2 - 2} e^{-\varphi^2(\xi)/2} d\xi$$

converges.

For the proof of this lemma, see Orey and Pruitt (1973, page 147).

2. Local Maxima. In this section we prove the main theorem dealing with the existence of the local maxima of the sample functions of the Bessel process B_t .

Theorem 2.1. For almost all sample functions of the Bessel process B_t defined in (1.1), there exists a local maxima.

Proof. Let s be the center of the unit interval U, and let $C_n \subset U$ be a cube with center at s, sides parallel to the coordinate axes and equal to a_n. Let u^n and v^n be the smallest and the largest vertex of C_n i.e. closest and farthest from the origin $\langle 0 \rangle$. Pick C_n with $\min(u_1^n, \dots, u_N^n) > \frac{1}{4}$.

Consider two points s^{nk} and v^{nk} of R_+^N determined by $s_k^{nk} = \frac{1}{2}$, $v_k^{nk} = v_k^n$, $s_j^{nk} = v_j^{nk} = u_j^n$ for $j \neq k$ where $1 \leq j \leq N$.

Let $\Delta(r,\ell)$ be any interval in U with at least two sides smaller than a_n. Define

$$A_{ni} = \bigcap_{k=1}^{N} [W^i(s^{nk}) - W^i(u) > 2a_n^{\frac{1}{2}},\ W^i(s^{nk}) - W^i(v^{nk}) > 2a_n^{\frac{1}{2}}]$$

$$B_{ni} = \bigcap_{k=1}^{N} [W^i(s^{nk}) - W^i(u) < -2a_n^{\frac{1}{2}},\ W^i(s^{nk}) - W^i(v^{nk}) < -2a_n^{\frac{1}{2}}]$$

$$C_{ni} = [\inf_{t \in C_n} W_t^i \geq 0],$$

$$E_{ni} = [\sup_{t \in C_n} W_t^i \leq 0],$$

$$F_{ni} = [\sup_{r,\ell \in U} |W^i(\Delta(r,\ell))| < (2^{N-1} - 1)^{-1} a_n^{\frac{1}{2}}]$$

The variables $W^i(s^{nk}) - W^i(u)$, $W^i(s^{nk}) - W^i(v^{nk})$ are normally distributed with mean 0 and variances greater than $4^{-N+\frac{1}{2}}a_n$. Thus $P(A_{ni}) > \beta$ for some constant β .

Let $\{a_n\}$ be a sequence of positive numbers with $a_n \downarrow 0$ and let D_n be the interior of $S(u^n, v^n)$. Clearly $D_n \downarrow \phi$ as $a_n \downarrow 0$. Observe that the event $[A_{ni}$ infinitely often$] \in \mathfrak{F}_\infty(D_n)$. Thus, from Lemma 1.1, it follows that

$$P(A_{ni} \text{ infinitely often}) = 1$$

Analogously,

$$P(B_{ni} \text{ infinitely often}) = 1 .$$

Let $\eta > 0$. Then by Lemma 1.2, or by the continuity of the sample functions of W ,

$$P[\bigcap_{n=n_0}^{\infty} C_{n_i} | W_s^i > \eta] \to 1 \quad \text{as} \quad n_0 \to \infty$$

Furthermore,

$$\begin{aligned} P[A_{ni}C_{ni} \text{ infinitely often}] &\geq P[(A_{ni} \text{ infinitely often}) \bigcap_{n=n_0}^{\infty} C_{ni}] . \\ &\geq P[\bigcap_{n=n_0}^{\infty} C_{ni}] \\ &\geq P[\bigcap_{n=n_0}^{\infty} C_{ni}[W_s^i > \eta]] \end{aligned}$$

Since $P[W_s^i > \eta]$ converges to $\frac{1}{2}$ as $\eta \to 0$, by picking η small enough and then n_0 large enough, the probability of the last event can be made as close to $\frac{1}{2}$ as desired.

It is now clear that

$$P[(A_{ni}C_{ni} \cup B_{ni}E_{ni}) \text{ i.o.}] = 1 .$$

Also, by Lemma 1.2 and by the independence of the components of W

$$P[\bigcap_{i=1}^{d} (A_{ni}C_{ni}F_{ni} \cup B_{ni}E_{ni}F_{ni}) \text{ i.o.}] = 1$$

Let $t \in \partial C_n$ where ∂C_n is the boundary of C_n, and let

$$p^t = \langle \sigma(t_1), \ldots, \sigma(t_N) \rangle$$

where

$$\sigma(t_i) = \begin{cases} t_i & \text{if } u_i^n < t_i < v_i^n \\ s_i & \text{otherwise} \end{cases}$$

Observe that p^t lies in the interior of C_n. Now consider events of the form

$$\bigcap_{i=1}^{d} \Lambda_{ni} \quad \text{where } \Lambda_{ni} = A_{ni}C_{ni}F_{ni} \quad \text{or} \quad B_{ni}E_{ni}F_{ni}$$

and note that there are 2^d events of this form.

We now claim that

$$(2.1) \qquad P[\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t \mid \bigcap_{i=1}^{d} A_{ni}C_{ni}F_{ni}] = 1$$

A slight variation of the proof along the same lines can be applied to $2^d - 1$ other events. To prove (2.1), let

$$G_n = \bigcap_{i=1}^{d} A_{ni} C_{ni} F_{ni} \tag{2.2}$$

$$H_n = \bigcap_{t \in \partial C_n} \{ \bigcap_{i=1}^{d} [W^i(p^t) - W^i(t) > a_n^{\frac{1}{2}}] \} .$$

Now

$$W^i(p^t) - W^i(t) = W^i(\langle \sigma(t_1), t_2, \ldots, t_N \rangle)$$

$$-W^i(\langle t_1, t_2, \ldots, t_N \rangle) + W^i(\langle \sigma(t_1), \sigma(t_2), t_3, \ldots, t_N \rangle)$$

$$-W^i(\langle \sigma(t_1), t_2, \ldots, t_N \rangle) + \ldots + W^i(\langle \sigma(t_1), \sigma(t_2), \ldots, \sigma(t_n) \rangle)$$

$$-W^i(\langle \sigma(t_1), \ldots, \sigma(t_{N-1}), t_N \rangle) .$$

Consider the random variable

$$W^i(\langle \sigma(t_1), \ldots, \sigma(t_{j-1}), \sigma(t_j), t_{j+1}, \ldots, t_N \rangle)$$

(2.3)

$$- W^i(\langle \sigma(t_1), \ldots, \sigma(t_{j-1}), t_j, t_{j+1} \ldots t_N \rangle).$$

The variance of this variable is equal to

$$(u_1^n + e_1^n) \ldots (u_{j-1}^n + e_{j-1}^n) |\sigma(t_j) - t_j| (u_{j+1}^n + e_{j+1}^n) \ldots (u_N^n + e_N^n)$$

where $0 \le e_i^n \le a_n$. It is now easy to see that (2.3) is equal to

$$W^i(\langle u_1^n, \ldots, u_{j-1}^n, \sigma(t_j), u_{j+1}^n, \ldots, u_N^n \rangle)$$

$$-W^i(\langle u_1^n, \ldots, u_{j-1}^n, t_j, u_{j+1}^n, \ldots, u_N^n\rangle) + L^i ,$$

where L^i can be decomposed into no more than $2^{N-1} - 1$ normally distributed random variables, such that the mean of each of these random variables is zero and the variance of each is equal to the N-dimensional Lebesgue measure of an interval in U with at least two sides smaller than a_n. Since $t \in \partial C_n$, t_j is equal to u_j^n or v_j^n and $\sigma(t_j) = \frac{1}{2}$ for some $1 \leq j \leq N$. Therefore

$$P(H_n \bigcap_{i=1}^{d} C_{ni} \mid G_n) = 1$$

However,

$$P[\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t \mid H_n \bigcap_{i=1}^{d} C_{ni}] = 1$$

Hence

$$P[\sup_{t \in C_n^0} B_t > \sup_{t \in \partial C_n} B_t \mid G_n] = 1$$

where G_n is defined in (2.2).

The proof is now completed. Recall that s was picked to be the center of U. Actually, s can be chosen to be any point in U^0. Therefore, for almost all sample functions of B_t, the set of local maxima is dense in R_+^N.

We shall now investigate some properties of the local maxima of B_t.

Definition 2.1. The sample function $B(\cdot,\omega)$ has a strict local maxima if there exists an open set 0 containing s such that $0 \subset R_+^N$ and $B(t,\omega) < B(s,\omega)$ for all $t \in 0$.

We have the following theorem.

Theorem 2.2. *For almost every sample function of* $\{B_t,\ t \in R_+^N\}$, *all the local maxima are strict and the set of local maxima is countable.*

Proof. Let I and J be two disjoint, closed intervals in the interior of R_+^N . We claim that

$$P\{\sup_{t\in I} B_t = \sup_{t\in J} B_t\} = 0 \ . \tag{2.4}$$

Let $I = \Delta(u,v)$, $J = \Delta(s,t)$. Denote the complements of $\Delta(t)$ and $\Delta(v)$ by $[\Delta(t)]'$ and $[\Delta(v)]'$. Since I and J are disjoint intervals, it is clear that

$$[\Delta(u) \cap [\Delta(t)]'] \cup [\Delta(s) \cap [\Delta(v)]']$$

contains a nondegenerate interval, i.e., an interval with positive N-dimensional Lebesgue measure. Let I' be any such interval, and without loss of generality assume that

$$I' \subset \Delta(u) \cap [\Delta(t)]'$$

Consider now

$$P[\sup_{t\in I} B_t = \sup_{t\in J} B_t]$$

$$= P[\sup_{t\in I}[(W_t^1)^2 + \sum_{i=2}^{d} (W_t^i)^2]^{\frac{1}{2}} = \sup_{t\in J}[\sum_{i=1}^{d} (W_t^i)^2]^{\frac{1}{2}}] .$$

Let $W^1(I')$ be the increment of W^1 over I' . Since W^1 has independent increments, for $t \in I$, we can write

$$W_t^1 = W_t^1 - W^1(I') + W^1(I')$$

such that $W^1(I')$ is independent of $W_t^1 - W^1(I')$ for all $t \in I$. Also, $W^1(I')$ is independent of $\sup_{t\in J}[\sum_{i=1}^{d}(W_t^i)^2]$ since $I' \subset [\Delta(t)]'$.

Let $X = W^1(I')$, $Y_t = W_t^1 - W^1(I')$. For an arbitrary fixed w , consider

$$\text{(2.5)} \qquad P[\sup_{t\in I}\{(X + Y_t(w))^2 + \sum_{i=2}^{d} (W_t^i(w))^2\}^{\frac{1}{2}}$$

$$= \sup_{t\in J}\{\sum_{i=1}^{d} (W_t^i(w))^2\}^{\frac{1}{2}}] .$$

We shall now show that (2.5) equals zero for a fixed w . Consider the function $f(x)$, defined by

$$f(x) = \sup_{t\in I}\{(x + Y_t(w))^2 + \sum_{i=2}^{d} (W_t^i)(w))^2\}^{\frac{1}{2}} .$$

Note that $f(x)$ equals the supremum of the distance from the origin of the set D_x in R^d defined by

$$D_x = \{z \in R^d : z_1 = Y_t(\omega) + x, \ z_2 = W_t^2(\omega), \ \ldots, \ z_d = W_t^d(\omega) \text{ for some } t \in I\} .$$

It is now easy to see that as x varies from $-\infty$ to $+\infty$, the set D_x is translated along a vector parallel to a coordinate axis and so $f(x)$ decreases and then increases as x goes from $-\infty$ to $+\infty$. For a fixed ω, (2.5) equals

(2.6) $$P[X = f^{-1}(\sup_{t \in J} \sum_{i=1}^{d} (W_t^i(\omega))^2)]$$

where f^{-1} is the inverse of f. It is clear that there are almost 2 values of $f^{-1}(\sup_{t \in J} \sum_{i=1}^{d} (W_t^i(\omega))^2)$, and since X is normal random variable, (2.6) equals 0. Thus, for each fixed ω, (2.5) equals zero. The proof of (2.4) follows by integrating (2.5) over the probability space.

Consider the set $$\bigcap [\sup_{t \in I} B_t \neq \sup_{t \in J} B_t]$$

where the intersection is taken over all intervals I and J with rational least and largest vertices, i.e., the coordinates of u, v, s, t are all rational. This set contains the set of ω such that all local maxima of $B(\cdot, \omega)$ are strict. Clearly, this set has probablity one.

Countability of the set of local maxima is a consequence of the following Lemma.

Lemma 2.1. *Let* f *be a continuous, real valued function on* R_+^N *with all local maxima strict. Then* f *has countably many local maxima.*

This Lemma is a straightforward generalization of the univariate case, the proof of which can be found in Freedman (1971).

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It is proved that almos every sample function of the N-Parameter Bessel process has a local maxima. In addition, some properties related to the local maxima are investigated.

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